Intuitive Approximations in Discrete Renewal Theory
Part 1: regularly varying case

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Abstract

It is usually impossible to find explicit expressions for the renewal sequence. This paper presents a simple method to approximate the renewal sequence, which covers many of the known approximations. The paper uses the ideas of Mitov and Omey (2014).

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1. Introduction

Suppose that $X, X_1, X_2, \ldots$ are i.i.d. nonnegative integer-valued random variables with p.d.f. $p_k = P(X = k)$, $k \in \mathbb{N}_0$. The d.f. of X is given by $F(x) = P(X \leq x)$ and its tail is denoted by $\overline{F}(x) = 1 - F(x)$. Throughout the paper we assume that (p_k) is aperiodic: $\gcd\{k : p_k > 0\} = 1$. We also assume that $0 < \mu = E(X) < \infty$. For $n \in \mathbb{N}_0$, the partial sums S_n are given by $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Note that $P(S_n \leq x) = F^{\star n}(x)$, which is the n-fold convolution of F, i.e. $F^{\star 0} = 1_{[0,\infty)}$ and $F^{\star n} = F \star F^{\star(n-1)}$, where the convolution of two d.f. is defined by $F \star G(x) = \int_0^x F(x-y)dG(y)$. For briefness, we write $F^{\star n}(x)$ as $F_k^{\star n}$ in case x is a nonnegative integer k. Moreover, $P(S_n = k) = p_k^{*n}$, which is the n-fold convolution of (p_k), i.e. $p_k^{*0} = 1_{\{0\}}(k)$ and $p_k^{*n} = (p * p^{*(n-1)})_k$, where the convolution of two sequences (a_k) and (b_k) is defined by $(a * b)_k = \sum_{i=0}^k a_i b_{k-i}$. The generating function of X is $\hat{P}(z) = E(z^X)$, $|z| < 1$, and $\hat{P}(1) = 1$. The generating function of S_n is given by $\hat{P}^n(z)$. Since X has finite expectation, we have $\mu = \hat{P}'(1)$.

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Let X_e be a random variable, independent of X, that has the equilibrium distribution corresponding to X, i.e. $p_{e,k} = P(X_e = k) = \overline{F}_k/\mu$ for $k \in \mathbb{N}_0$. The generating function of X_e satisfies $\hat{P}_e(z) = (1-\hat{P}(z))/(\mu(1-z))$. Define $S_{e,n} = X_{e,1} + \cdots + X_{e,n}$, $p^{*n}_{e,k}$, and $F^{\star n}_{e,k}$ analogously as above.

The renewal sequence (u_n) is defined by $u_n = \sum_{k=0}^{\infty} p_n^{*k}$. The aim of the present paper is to obtain approximations for u_n when n is large. Therefore, all limits that appear later are taken with respect to $n \to \infty$. It is well known that $u_n \to 1/\mu$ and the main problem is to obtain precise estimates for the rate at which $u_n - 1/\mu \to 0$ or $\Delta u_n = u_{n-1} - u_n \to 0$.

In our approach, we start from the generating function of (u_n), which is given by $\hat{U}(z) = \sum_{n=0}^{\infty} u_n z^n = (1-\hat{P}(z))^{-1} = (\mu(1-z)(1-(1-\hat{P}_e(z))))^{-1}$. Using a Taylor expansion, we obtain that

$$\hat{U}(z) = \sum_{k=0}^{\infty} \hat{T}_k(z), \quad \text{with} \quad \hat{T}_k(z) = \frac{1}{\mu(1-z)}(1-\hat{P}_e(z))^k. \tag{1}$$

Formula (1) suggests the following approximations $\hat{U}_m(z)$ for $\hat{U}(z)$: $\hat{U}_m(z) = \sum_{k=0}^{m} \hat{T}_k(z)$. By inversion, this approach then leads to approximations $u_{m,n}$ for the renewal sequence u_n of the form $u_{m,n} = \sum_{k=0}^{m} t_{k,n}$, where the sequence $(t_{k,n})$ has generating function $\hat{T}_k(z) = \sum_{n=0}^{\infty} t_{k,n} z^n$. In the next section we will identify $\hat{T}_k(z)$ and $(t_{k,n})$. In this paper we focus on the cases $0 \le m \le 3$ and show that our approximations $(u_{m,n})$ correspond to the approximations that have been published in many papers before.

2. The sequences $(t_{k,n})$ and $(\delta_{k,n})$

2.1. Expressions for $(t_{k,n})$ and $(\delta_{k,n})$

We first identify $\hat{T}_k(z)$. If $k = 0$, then (1) gives $\hat{T}_0(z) = 1/(\mu(1-z))$, which shows that $\hat{T}_0(z)$ is the generating function of $t_{0,n} = 1/\mu$. For $k \ge 1$, the binomial expansion in (1) yields

$$\begin{aligned}\hat{T}_k(z) &= \frac{1}{\mu(1-z)} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \hat{P}_e^i(z) \\ &= \frac{-1}{\mu(1-z)} \sum_{i=1}^{k} \binom{k}{i} (-1)^i (1-\hat{P}_e^i(z)).\end{aligned}$$

Since $\hat{P}_e^i(z)$ is the generating function of $S_{e,i}$, we have $(1-\hat{P}_e^i(z))/(1-z) = \sum_{n=0}^{\infty} \overline{F^{\star i}_{e,n}} z^n$. We therefore obtain the following result.

Lemma 1. *For $n \geq 1$, let $\delta_{k,n} = t_{k,n-1} - t_{k,n}$. Then $t_{0,n} = 1/\mu$, $\delta_{0,n} = 0$ and, if $k \geq 1$,*

$$t_{k,n} = -\frac{1}{\mu} \sum_{i=1}^{k} \binom{k}{i} (-1)^i \overline{F_{e,n}^{\star i}}, \tag{2}$$

$$\delta_{k,n} = -\frac{1}{\mu} \sum_{i=1}^{k} \binom{k}{i} (-1)^i p_{e,n}^{*i}. \tag{3}$$

We now consider into detail the cases $k = 1, 2, 3$.

Lemma 2. *For $k \geq 2$, let $R_{k,n}^e = \overline{F_{e,n}^{\star k}} - k\overline{F}_{e,n}$ and $r_{k,n}^e = p_{e,n}^{*k} - kp_{e,n}$. Then*

$$\begin{aligned} t_{1,n} &= \overline{F}_{e,n}/\mu & \qquad \delta_{1,n} &= p_{e,n}/\mu \\ t_{2,n} &= -R_{2,n}^e/\mu & \delta_{2,n} &= -r_{2,n}^e/\mu \\ t_{3,n} &= (R_{3,n}^e - 3R_{2,n}^e)/\mu & \delta_{3,n} &= (r_{3,n}^e - 3r_{2,n}^e)/\mu \end{aligned}$$

Proof. The results follow directly from (2) and (3). □

2.2. Asymptotic behaviour of $(t_{k,n})$ and $(\delta_{k,n})$, $1 \leq k \leq 3$.

In order to discuss the asymptotic behaviour of $(t_{k,n})$ and $(\delta_{k,n})$, we recall some basic definitions and properties of regularly varying sequences.

2.2.1. Regularly varying sequences

A sequence of real numbers (a_n) is regularly varying at infinity and with real index α if $a_n > 0$ for n large and if

$$\lim_{x\to\infty} \frac{a_{[xy]}}{a_{[x]}} = y^\alpha, \quad \forall y > 0. \tag{4}$$

Notation: $(a_n) \in RS(\alpha)$. We write $(a_n) \in RS$ if $(a_n) \in RS(\alpha)$ for some $\alpha \in \mathbb{R}$. If $(a_n) \in RS(\alpha)$, then (4) holds locally uniformly in $y > 0$; see sections 1.2 and 1.9 in Bingham et al. (1987). From this it follows that $RS \subset LS$, where a sequence of real numbers (a_n) is in the class LS if $a_n > 0$ for n large and if $a_{n+1}/a_n \to 1$. If $(a_n) \in RS(\alpha)$, then for each $\epsilon > 0$ we can find constants A, B, x° so that

$$Ay^{\alpha-\epsilon} \leq \frac{a_{[xy]}}{a_{[x]}} \leq By^{\alpha+\epsilon}, \quad \forall x \geq x^\circ, \forall y \geq 1,$$

cf. section 1.5 in Bingham et al. (1987) or Proposition 1.7 in Geluk and de Haan (1987). For sequences (a_n) and (b_n), we use the following notations:

- $a_n = O(1)b_n$ means that $\limsup_{n\to\infty} a_n/b_n < \infty$;
- $a_n = o(1)b_n$ means that $\limsup_{n\to\infty} a_n/b_n = 0$;
- $a_n \approx b_n$ means that $a_n = O(1)b_n$ and $b_n = O(1)a_n$;
- $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$.

Note that for $(a_n) \in RS(\alpha)$ we have $a_n = o(1)n^{\alpha+\epsilon}$. If $\alpha + \beta < -1$, then Karamata's theorem states that $\sum_{n=0}^{\infty} n^{\beta} a_n < \infty$ and

$$\sum_{k=n}^{\infty} k^{\beta} a_k \sim -\frac{n^{\beta+1}a_n}{\alpha+\beta+1}, \tag{5}$$

cf. section 1.6 in Bingham et al. (1987). Using local uniform convergence, it also follows that for $B \geq A > 0$ we have

$$\sum_{k=[An]}^{[Bn]} k^{\beta} a_k \approx n^{\beta+1} a_n. \tag{6}$$

The next lemma will be used in the sections below.

Lemma 3. *Suppose that $(a_n) \in RS(-\alpha)$ and $(b_n) \in RS$.*

(i) If $\alpha > 1$, then $\sum_{k=0}^{[n/2]} b_{n-k}a_k/b_n \to \sum_{k=0}^{\infty} a_k$;
(ii) If $\alpha > 2$, then $\sum_{k=0}^{[n/2]} \sum_{i=0}^{k} b_{n-i}a_k/b_n \to \sum_{k=0}^{\infty} ka_k$.

Proof. Note that $\alpha > 1$ implies that $\sum_{k=0}^{\infty} a_k < \infty$ and that $\alpha > 2$ implies that $\sum_{k=0}^{\infty} ka_k < \infty$.

(i) Because $(b_n) \in LS$, we have $b_{n-k}/b_n \to 1$ for all k. Since $0 \leq k \leq [n/2]$, $[n/2] \leq n - [n/2] \leq n - k \leq n$. From $(b_n) \in RS$ now follows that

$$\frac{b_{n-k}}{b_n} \leq \sup_{1/2 \leq x \leq 1} \frac{b_{[nx]}}{b_n} \leq C, \quad n \geq n^{\circ}.$$

We can use Lebesgue's theorem to conclude that

$$\frac{1}{b_n} \sum_{k=0}^{[n/2]} b_{n-k} a_k \to \sum_{k=0}^{\infty} a_k.$$

(ii) As in (i) we have $b_{n-i}/b_n \to 1$, for all i, and for each fixed k we have $\sum_{i=0}^{k} b_{n-i}/b_n \to k$. Now we have

$$\frac{1}{b_n}\sum_{i=0}^{k} b_{n-i} \leq \sup_{n-k\leq j\leq n} \frac{b_j}{b_n} k \leq Ck, \quad n \geq n^\circ.$$

We can again use Lebesgue's theorem to obtain result (ii). □

2.2.2. Asymptotic behaviour of $(t_{1,n})$ and $(\delta_{1,n})$
Recall from Lemma 2 that $t_{1,n} = \overline{F}_{e,n}/\mu$ and $\delta_{1,n} = p_{e,n}/\mu = \overline{F}_n/\mu^2$. We now apply Karamata's theorem, cf. (5), to obtain the following result.

Lemma 4. *(i) If $(\overline{F}_n) \in RS(-\alpha)$ and $\alpha > 1$, then $\mu < \infty$, $\delta_{1,n} = \overline{F}_n/\mu^2$, and $t_{1,n} \sim n\overline{F}_n/(\mu^2(\alpha-1))$.*
(ii) If $(p_n) \in RS(-\alpha)$ and $\alpha > 2$, then $\mu < \infty$, $\overline{F}_n \sim np_n/(\alpha-1)$, $\delta_{1,n} \sim np_n/(\mu^2(\alpha-1))$, and $t_{1,n} \sim n^2 p_n/(\mu^2(\alpha-1)(\alpha-2))$.

Remark 1. From $t_{1,n} = \overline{F}_{e,n}/\mu$ and $\delta_{1,n} = \overline{F}_n/\mu^2$ (cf. Lemma 2), we also obtain that

$$\sum_{n=1}^{\infty} n^r t_{1,n} = \frac{1}{\mu}\sum_{n=1}^{\infty} n^r \overline{F}_{e,n} \leq CE(X_e^{r+1}),$$
$$\sum_{n=1}^{\infty} n^r \delta_{1,n} = \frac{1}{\mu^2}\sum_{n=1}^{\infty} n^r \overline{F}_n \leq CE(X^{r+1}).$$

2.2.3. Asymptotic behaviour of $(t_{2,n})$ and $(\delta_{2,n})$
Recall from Lemma 2 that $t_{2,n} = -R^e_{2,n}/\mu$ and $\delta_{2,n} = -r^e_{2,n}/\mu$, where $R^e_{2,n} = P(X_{e,1}+X_{e,2} > n) - 2P(X_e > n)$ and $r^e_{2,k} = P(X_{e,1}+X_{e,2} = n) - 2P(X_e = n)$, with $X_{e,1}$ and $X_{e,2}$ i.i.d. copies of X_e.

Two lemmas. In the first result we study more generally the asymptotic behaviour of $R_{2,n}$ and $r_{2,n}$, where

$$R_{2,n} = P(X+Y > n) - P(X > n) - P(Y > n),$$
$$r_{2,n} = P(X+Y = n) - P(X = n) - P(Y = n),$$

and where X and Y are independent discrete r.v. with $a_n = P(X = n)$ and $b_n = P(Y = n)$, $n \geq 0$. For further use, starting from (a_n), we define

$\delta_n^a = a_{n-1} - a_n$, $n \geq 1$. Clearly, for $j > i$, we have

$$\sum_{n=i+1}^{j} \delta_n^a = a_i - a_j \quad \text{and} \quad \sum_{n=i+1}^{\infty} \delta_n^a = a_i.$$

We use a similar notation for (b_n). Note that $R_{2,n} = \sum_{k=n+1}^{\infty} r_{2,k}$ and that $P(X+Y=n) = (a*b)_n$. First we consider $R_{2,n}$ and $(a*b)_n$.

Lemma 5. *Assume that $(a_n) \in RS(-\alpha)$ and $(b_n) \in RS(-\beta)$, with $\alpha > 2$ and $\beta > 2$.*

(i) We have $E(X) + E(Y) < \infty$ and

$$R_{2,n} = b_n E(X) + a_n E(Y) + o(1)a_n + o(1)b_n;$$

(ii) $(a*b)_n = a_n + b_n + o(1)a_n + o(1)b_n$.

Proof. Let $m = [n/2]$.

(i) We have $R_{2,n} = I + II + III - IV$, where

$$\begin{aligned} I &= P(X+Y>n, m<Y\leq n, X\leq m) \\ II &= P(X+Y>n, m<X\leq n, Y\leq m) \\ III &= P(m<X\leq n, m<Y\leq n) \\ IV &= P(X>n, Y>n). \end{aligned}$$

First we consider I. We clearly have

$$I = \sum_{k=0}^{m} P(n-k<Y\leq n)P(X=k) = \sum_{k=0}^{m}\sum_{i=0}^{k-1} b_{n-i}a_k.$$

Using Lemma 3 (ii) we obtain that $I/b_n \to \sum_{k=0}^{\infty} k a_k = E(X)$. In a similar way, we obtain that $II/a_n \to E(Y)$. Now consider III. Since $m = [n/2]$ and $(a_n), (b_n) \in RS$, we have, cf. (6),

$$III = \sum_{i=m+1}^{n} a_i \sum_{j=m+1}^{n} b_j \approx n^2 a_n b_n.$$

Since $\alpha > 2$, it follows that $n^2 a_n \to 0$ and $III = o(1)b_n$. Finally, for IV we have $P(X>n) \approx n a_n$ and $P(Y>n) \approx n b_n$, cf. (5), and it therefore follows that $IV \approx n^2 a_n b_n = o(1)a_n$. Combining the estimates for I, II, III, and IV, result (i) follows.

(ii) We have

$$(a*b)_n = \sum_{k=0}^{m} a_k b_{n-k} + \sum_{k=0}^{n-m-1} a_{n-k} b_k = I + II.$$

Using Lemma 3 (i) we have $I/b_n \to \sum_{k=0}^{\infty} a_k = 1$ and $II/a_n \to 1$. $\square$

In our next result we discuss $r_{2,n}$.

Lemma 6. *Suppose that $(\delta_n^a) \in RS(-\alpha)$ and $(\delta_n^b) \in RS(-\beta)$, with $\alpha > 3$ and $\beta > 3$. Then $r_{2,n} = \delta_n^b E(X) + \delta_n^a E(Y) + o(\delta_n^a) + o(\delta_n^b)$.*

Proof. Using $m = [n/2]$, we have $r_{2,n} = I + II$, where

$$I = \sum_{k=0}^{m} a_k b_{n-k} - b_n \quad \text{and} \quad II = \sum_{k=0}^{n-m-1} a_{n-k} b_k - a_n.$$

First consider I. We have

$$I = \sum_{k=1}^{m} a_k (b_{n-k} - b_n) - b_n \sum_{k=m+1}^{\infty} a_k = I_A - I_B.$$

Using $b_{n-k} - b_n = \sum_{i=0}^{k-1} \delta_{n-i}^b$, we have $I_A = \sum_{k=1}^{m} a_k \sum_{i=0}^{k-1} \delta_{n-i}^b$. Lemma 3 (ii) shows that $I_A/\delta_n^b \to E(X)$. Now consider I_B. Using $\sum_{k=n+1}^{\infty} \delta_k^b = b_n$, $(\delta_n^b) \in RS(-\beta)$ with $\beta > 3$, we have $b_n \approx n\delta_n^b$ and $(b_n) \in RS(1-\alpha)$. Similarly, $a_n \approx n\delta_n^a$ and $(a_n) \in RS(1-\alpha)$. Since $\alpha - 1 > 2$, it follows that $\sum_{k=n+1}^{\infty} a_k \approx n a_n \approx n^2 \delta_n^a$. Combining these estimates, it follows that

$$I_B \approx n\delta_n^b \sum_{k=m+1}^{\infty} a_k \approx n\delta_n^b m^2 \delta_m^a.$$

Since $m = [n/2]$, $(\delta_n^a) \in RS$, and $\beta > 3$, we conclude that $I_B \approx n^3 \delta_n^b \delta_n^a = o(1)\delta_n^a$. We similarly find that $II/\delta_n^a \to E(Y)$. $\square$

Asymptotic behaviour of $(t_{2,n})$ *and* $(\delta_{2,n})$. We apply Lemma 5 and Lemma 6, with $a_n = b_n = p_{e,n}$ and $\delta_n^a = p_{e,n-1} - p_{e,n} = (\overline{F}_{n-1} - \overline{F}_n)/\mu = p_n/\mu$.

Theorem 7. *(i) If $(p_{e,n}) \in RS(-\alpha)$, with $\alpha > 2$, then $t_{2,n} \sim -2\mu_e p_{e,n}/\mu$.*

(ii) If $(p_n) \in RS(-\alpha)$, with $\alpha > 3$, then $p_{e,n} \sim np_n/(\mu(\alpha-1))$, $\delta_{2,n} \sim -2\mu_e p_n/\mu^2$, and $t_{2,n} \sim -2np_n\mu_e/(\mu^2(\alpha-1))$.

Proof. Recall from Lemma 2 that $t_{2,n} = -R^e_{2,n}/\mu$ and $\delta_{2,n} = -r^e_{2,n}/\mu$.

(i) Lemma 5 (i) shows that $R^e_{2,n} \sim 2\mu_e p_{e,n}$.

(ii) $(p_n) \in RS(-\alpha)$, $\alpha > 3$, and (5) yield $p_{e,n} = \overline{F}_n/\mu \sim np_n/(\mu(\alpha-1))$. This also shows that $(p_{e,n}) \in RS(-(\alpha-1))$. The result for $t_{2,n}$ thus follows from (i). Since $(\delta^a_n) = (p_n/\mu) \in RS(-\alpha)$ and $\alpha > 3$, Lemma 6 shows that $\delta_{2,n} = -r^e_{2,n}/\mu \sim -2\delta^a_n\mu_e/\mu = -2p_n\mu_e/\mu^2$. □

Remark 2. Note that

$$\sum_{n=1}^{\infty} n^r |t_{2,n}| \leq C \sum_{n=1}^{\infty} n^r \left|R^e_{2,n}\right| \leq C \left(E(X_e+Y_e)^{r+1} + EX_e^{r+1}\right).$$

2.2.4. Asymptotic behaviour of $(t_{3,n})$
Recall from Lemma 2 that $t_{3,n} = (R^e_{3,n} - 3R^e_{2,n})/\mu$.

One lemma. We first discuss, more generally, the asymptotic behaviour of $A_n = R_{3,n} - 3R_{2,n}$, where $R_{2,n} = P(X+Y>n) - 2P(X>n)$, $R_{3,n} = P(X+Y+Z>n) - 3P(X>n)$, and X, Y and Z are i.i.d. with $a_n = P(X=n)$. We now use $\delta_n = a_{n-1} - a_n$.

Lemma 8. *If* $(\delta_n) \in RS(-\alpha)$ *with* $\alpha > 3$, *then* $A_n/\delta_n \to 3\,(E(X))^2$.

Proof. Direct calculations show that

$$A_n = \sum_{k=0}^{n} R_{2,n-k}a_k - R_{2,n} = I + II,$$

where (using $m = [n/2]$)

$$I = \sum_{k=0}^{m} R_{2,n-k}a_k - R_{2,n} \quad \text{and} \quad II = \sum_{k=m+1}^{n} R_{2,n-k}a_k.$$

Clearly,

$$I = \sum_{k=0}^{m}(R_{2,n-k} - R_{2,n})a_k - R_{2,n}\sum_{k=m+1}^{\infty} a_k = I_A - I_B$$

and

$$II = \sum_{k=0}^{n-m-1} R_{2,k}(a_{n-k} - a_n) + a_n \sum_{k=0}^{n-m-1} R_{2,k} = II_A + II_B.$$

First consider I_A. We have $R_{2,n-k} - R_{2,n} = \sum_{j=0}^{k-1} r_{2,n-j}$. Lemma 6 yields $r_{2,n} \sim 2\delta_n E(X) \in RS(-\alpha)$. Lemma 3 (ii) shows that $I_A/\delta_n \to 2(E(X))^2$. Next consider I_B. Lemma 5 (i) shows that $R_{2,n} \approx a_n$. Since $(a_n) \in RV(1-\alpha)$, it follows that $I_B \approx a_n \sum_{k=m}^{\infty} a_k \approx na_n^2$. Since $a_n \approx n\delta_n$ and $\alpha > 3$, we obtain that $I_B = o(\delta_n)$.

Now consider II_A. Using $a_{n-k} - a_n = \sum_{i=0}^{k-1} \delta_{n-i}$ and $(\delta_n) \in RS(-\alpha) \subset LS$, Lemma 3 (ii) gives $II_A/\delta_n \to \sum_{k=0}^{\infty} kR_{2,k}$. Remark 3 below shows that $\sum_{k=0}^{\infty} kR_{2,k} = (E(X))^2$. Finally consider II_B. Since $\sum_{n=0}^{\infty} R_{2,n} = 0$, we have $II_B = -a_n \sum_{k=n-m}^{\infty} R_{2,k}$. Using $R_{2,n} \approx a_n$, cf. Lemma 5, and $(a_n) \in RV(1-\alpha)$ with $\alpha > 3$, we have $II_B/\delta_n \approx na_n^2/\delta_n \approx n^3\delta_n \to 0$. □

Remark 3. For the sequence $(R_{2,n})$ we have

$$\hat{R}_2(z) = \sum_{n=0}^{\infty} z^k R_{2,k} = -\frac{(1-\hat{P}(z))^2}{1-z}.$$

If $\mu < \infty$, we have

$$\lim_{z\to 1} \hat{R}_2(z) = -\lim_{z\to 1} \frac{1-\hat{P}(z)}{1-z}(1-\hat{P}(z)) = -\mu \times 0 = 0,$$

or $\hat{R}_2(1) = 0$. Using this result, we get

$$\hat{R}_2'(1) = \lim_{z\to 1} \frac{\hat{R}_2(1) - \hat{R}_2(z)}{1-z} = \lim_{z\to 1} \frac{(1-\hat{P}(z))^2}{(1-z)^2} = \mu^2.$$

Asymptotic behaviour of $(t_{3,n})$. Applying Lemma 8 gives the following result.

Theorem 9. *If* $(p_n) \in RS(-\alpha)$ *with* $\alpha > 3$, *then* $t_{3,n}/p_n \to 3(\mu_e/\mu)^2$.

3. Consecutive approximations for (u_n)

Since $u_{m,n} = \sum_{k=0}^{m} t_{k,n}$ and $\Delta u_{m,n} = u_{m,n-1} - u_{m,n} = \sum_{k=1}^{m} \delta_{k,n}$, Lemma 2 yields the following approximations for u_n and $\Delta u_n = u_{n-1} - u_n$:

$$u_{0,n} = \frac{1}{\mu}, \qquad \Delta u_{0,n} = 0, \tag{7}$$

$$u_{1,n} = u_{0,n} + \frac{1}{\mu}\overline{F}_{e,n}, \qquad \Delta u_{1,n} = \frac{1}{\mu} p_{e,n}, \tag{8}$$

$$u_{2,n} = u_{1,n} - \frac{1}{\mu} R^e_{2,n}, \qquad \Delta u_{2,n} = \frac{1}{\mu} p_{e,n} - \frac{1}{\mu} r^e_{2,n}, \tag{9}$$

$$u_{3,n} = u_{2,n} + \frac{1}{\mu}(R^e_{3,n} - 3R^e_{2,n}). \tag{10}$$

3.1. The approximation $(u_{0,n})$

Blackwell's theorem states that the renewal sequence satisfies $u_n \to 1/\mu$ and $\Delta u_n \to 0$. In our case, (7) shows that $u_{0,n} = 1/\mu$ and $\Delta u_{0,n} = 0$.

3.2. The approximation $(u_{1,n})$

If $(\overline{F}_n) \in RS(-\alpha)$ and $\alpha > 1$, then (8) and Lemma 4 show that

$$u_{1,n} - \frac{1}{\mu} = \frac{\overline{F}_{e,n}}{\mu} \sim \frac{n\overline{F}_n}{\mu^2(\alpha - 1)}$$

and $\Delta u_{1,n} = \overline{F}_n/\mu^2$. In renewal theory, the following results are known, cf. Theorems 3.1.6 and 3.1.7 in Frenk (1983):

- If $(\overline{F}_n) \in RS(-\alpha)$ with $\alpha > 1$, then

$$u_n - \frac{1}{\mu} \sim \frac{\overline{F}_{e,n}}{\mu} \sim \frac{n\overline{F}_n}{\mu^2(\alpha - 1)}.$$

- As in Remark 1, several authors have discussed moment conditions on X or X_e that imply $\sum_{n=1}^{\infty} n^r |u_n - 1/\mu| < \infty$ or $\sum_{n=1}^{\infty} n^r |\Delta u_n| < \infty$. We refer to Feller (1949), Karlin (1955), Stone (1965), Stone and Wainger (1967), and Grübel (1979) for such type of results.

3.3. The approximation $(u_{2,n})$

If $(p_n) \in RS(-\alpha)$ with $\alpha > 3$, then (9) and Theorem 7 show that

$$u_{2,n} - \frac{1}{\mu} - \frac{\overline{F}_{e,n}}{\mu} = -\frac{R^e_{2,n}}{\mu} \sim -\frac{2\mu_e}{\mu^2(\alpha - 1)} np_n,$$

and

$$\Delta u_{2,n} - \frac{p_{e,n}}{\mu} = -\frac{r^e_{2,n}}{\mu} \sim -\frac{2\mu_e}{\mu^2} p_n.$$

From Remark 2, it follows that $E(X_e^{r+1}) < \infty$ implies that

$$\sum_{n=1}^{\infty} n^r \left| u_{2,n} - \frac{1}{\mu} - \frac{1}{\mu}\overline{F}_{e,n} \right| < \infty.$$

The following results are known in renewal theory:

- Theorem 3.1.12 of Frenk (1983) and Theorem 2.1.25 of Frenk (1987) provide conditions under which

$$u_n - \frac{1}{\mu} - \frac{1}{\mu}\overline{F}_{e,n} \sim -\frac{2\mu_e}{\mu^2}\overline{F}_n.$$

  See also Rogozin (1973).

- Theorem 3.1.14 of Frenk (1983) and Theorem 2.1.16 of Frenk (1987) also consider the case where $(p_{e,n}) \in RS(-\alpha)$ with $1 < \alpha < 2$.

- Theorem 2.1.20 in Frenk (1987) provides a sufficient condition of the type $E(X^{r+1}) < \infty$ for

$$\sum_{n=1}^{\infty} n^r \left| u_n - \frac{1}{\mu} - \frac{1}{\mu}\overline{F}_{e,n} \right| < \infty.$$

3.4. The approximation $(u_{3,n})$

For $(p_n) \in RS(-\alpha)$ with $\alpha > 3$, (10) and Theorem 9 lead to the following approximation:

$$u_{3,n} = \frac{1}{\mu} + \frac{1}{\mu}\overline{F}_{e,n} - \frac{1}{\mu}R^e_{2,n} + \frac{3\mu_e^2}{\mu^2}p_n + o(1)p_n.$$

We have found no similar result for the renewal sequence in the renewal literature. The approximation $u_{3,n}$ implicitly appears in Section 4 of Grübel (1983). See also Essén (1973) and Ney (1981).

4. Concluding remarks

1. In this paper we studied the renewal sequence when p_n is regularly varying and thus bounded by a power of n. This approach does not cover exponential cases such as $p_n = Ce^{-n}$ or $p_n \sim n^\alpha c^n$, $0 < c < 1$. In a forthcoming paper we will also study this case.
2. In many cases the condition $(a_n) \in RS(-\alpha)$ can be replaced by the condition $(a_n) \in ORS \cap LS$, where ORS is the class of sequences (a_n) for which (4) is replaced by $\forall y > 0 : a_{[xy]} = O(a_{[x]})$ as $x \to \infty$.
3. To study the asymptotic behaviour of $\delta_{3,n}$ we should impose conditions on $p_n - p_{n-1}$. The details are somewhat complicated and further research is needed here.

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